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Variable-range hopping in 2D quasi-1D electronic systems

S. Teber[a]

The Abdus Salam ICTP, Strada Costiera 11, 34014, Trieste, Italy

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Abstract. A semi-phenomenological theory of variable-range hopping (VRH) is developed for two-dimensional (2D) quasi-one-dimensional (quasi-1D) systems such as arrays of quantum wires in the Wigner crystal regime. The theory follows the phenomenology of Efros, Mott and Shklovskii allied with microscopic arguments. We first derive the Coulomb gap in the single-particle density of states, $g(\varepsilon)$, where ε is the energy of the charge excitation. We then derive the main exponential dependence of the electron conductivity in the linear (L), i.e. $\sigma(T) \sim \exp[-(T_L/T)^{\gamma_L}]$, and current in the non-linear (NL), i.e. $j(\mathcal{E}) \sim \exp[-(\mathcal{E}_{NL}/\mathcal{E})^{\gamma_{NL}}]$, response regimes ($\mathcal{E}$ is the applied electric field). Due to the strong anisotropy of the system and its peculiar dielectric properties we show that unusual, with respect to known results, Coulomb gaps open followed by unusual VRH laws, i.e. with respect to the disorder-dependence of T_L and $\mathcal{E}_{NL}$ and the values of γ_L and γ_{NL}.

PACS. 73.50.Bk General theory, scattering mechanisms – 72.20.Ee Mobility edges; hopping transport – 71.45.Lr Charge-density-wave systems

1 Introduction

Issues related to transport in low-dimensional electronic systems are challenging, the main difficulties being to take into account of disorder or interactions and in some cases of the interplay between them. As known from Mott and Towse [1], for strictly one-dimensional (1D) systems, any disorder leads to localized states. Such a statement implies that localization and transport properties of 1D Anderson insulators may be tackled perturbatively by considering the limit of a weak disorder (so called Gaussian disorder where the density of impurities $N \to \infty$, their strength $W \to 0$ while NW^2 is constant). For such a weak disorder, and in the absence of electron-electron and electron-phonon interactions, Berezinskii [2] has confirmed the statement of Mott and showed that the a.c. conductivity is given by: $\sigma_{ac}(\omega) \sim \omega^2 \log^2 \omega$. Subsequently, his approach has been extended by Gogolin, Mel'nikov and Rashba (GMR) [3] to the case where an electron-phonon coupling is present. They have shown that three-dimensional (3D) phonons provide a delocalization mechanism for the electrons at temperatures low enough that the scattering is mainly elastic, e.g. $\tau_{in}(T) \gg \tau_{el}$ where $\tau_{in}(T)$ is the inelastic phonon scattering time and τ_{el} is the elastic scattering time on the static defects. This delocalization mechanism leads to a *power-law hopping* for the d.c. conductivity, e.g. $\sigma_0(T) \sim T^3$, where the power comes from the phonon-scattering time. It should be noted that their arguments are not valid at the lowest temperatures; in particular, they hold only above $T_0 \propto 1/\tau_{el}$, here and below: $\hbar = 1$ unless specified. Subsequent studies aimed at exploring the effect of electron-electron interactions in 1D Anderson insulators. In this respect, it has been shown [4] that, in a Luttinger liquid, the Gaussian disorder is strongly renormalized by interactions. For repulsive interactions, each impurity becomes effectively strong. The 1D disordered interacting system is then equivalent [5] to an ensemble of weak links where impurities act as wire breakers. As a consequence, the power-law hopping laws acquire a non-universal exponent, e.g. interaction-parameter dependent. More recently [6], the low-temperature situation where the coupling to phonons is absent and electron-electron interactions dominate has been addressed. The modern notion of dephasing, due to these electron-electron interactions, has been considered as the delocalization mechanism of the electrons. It leads to a power-law hopping regime, reminiscent of the results of GMR, followed by a drastic suppression of the conductivity [7]. Less explored, from microscopic techniques, is the low-temperature regime: $T \ll T_0 \propto 1/\tau_{el}$, with both electron-electron and electron-phonon couplings. In this case, it is generally believed that the transport is of the *variable-range hopping* (VRH) type. In systems with 3D phonons, that we shall be concerned with in the rest of this manuscript, semi-phenomenological arguments by Mott [8] suggest that the VRH laws are

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[a] e-mail: `steber@ictp.trieste.it`

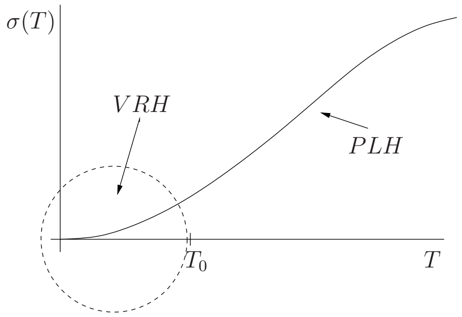

Fig. 1. d.c. conductivity as a function of temperature for a quasi-one dimensional system coupled to three-dimensional phonons. In the range of temperatures displayed, phonons provide a delocalization mechanism for the charge excitations of the system. Power-law hopping (PLH) dominates at temperatures $T \gg T_0$ and variable-range hopping (VRH) dominates at $T \ll T_0$ ($T_0 \propto 1/\tau_{el}$ where τ_{el} is the elastic scattering time on the static defects).

characterized by stretched exponentials and read:

$$\sigma(T) = \sigma_0(T) \exp[-(T_0/T)^{\gamma}], \tag{1}$$

where $\gamma \leq 1$. In 1D, $T_0 \propto 1/\nu\xi$, where the single-particle density of states $\nu \propto 1/v_F$ and the localization length $\xi \propto v_F \tau_{el}$. Hence, $T_0 \propto 1/\tau_{el}$, and the exponential dependence manifests at $T \ll T_0 \propto 1/\tau_{el}$. In the opposite case, the temperature dependence of the conductivity arises mainly from the pre-factor: $\sigma_0(T)$, which depends on the electron-phonon coupling, and leads to power-law hopping, see Figure 1 for a summary of regimes. The success of Mott's arguments came from their wide experimental confirmation for a great variety of disordered systems, i.e. it has been found experimentally that $\gamma \approx 0.25$ in isotropic 3D and 0.33 in isotropic 2D systems, in accordance with the theory. Later, Efros and Shklovskii (ES), see reference [9] for a review, extended these arguments to the Mott-Anderson insulators where, besides strong disorder, the long-range Coulomb interaction is present. Once again, their results were confirmed by a great variety of experiments in the field of doped semiconductors where the exponent $\gamma \approx 0.5$ in all dimensions [10] as predicted by ES.

In the present study, we focus on the low-temperature transport properties of strongly disordered and interacting two-dimensional (2D) quasi-1D electron systems. The latter consist of a 2D periodic lattice of parallel wires and therefore display markedly anisotropic properties with relations to the 1D world, see references [11] and [12] for reviews on 1D physics. We assume that impurities are point-like and act as wire breakers. Interactions are long-ranged and lead to a large parameter: $r_s = U_H/\varepsilon_F$, corresponding to the ratio of the Coulomb energy-scale: $U_H = e^2/\kappa a$ (κ is the dielectric constant of the host lattice in which the 2D system is embedded), to the kinetic energy scale given by the Fermi energy: ε_F. In the field of charge-density waves (CDW) [13], such quasi-1D Mott insulators are referred to as $4k_F$ CDWs. This $4k_F$ modulation corresponds to a space periodicity of the charge-density along the wires, $\lambda_{4k_F} = 2\pi/4k_F$, equal to the average distance between electrons a; such systems therefore display a strongly correlated state of the Wigner crystal type. For our purposes, such systems include man-made atomic or molecular wire arrays embedded in a semi-conducting matrix [15] and mono-layers of CDWs [16–18]. For these quasi-1D Mott-Anderson insulators, where the interplay between disorder and interactions is quite non-trivial and out of the reach of any perturbative scheme, we will naturally [19] follow the semi-phenomenological route of Mott and Efros and Shklovskii in deriving the low-T (i.e. $T \ll 1/\tau_{el}$) [21] transport properties. Our motivation resides in the existing experimental literature on quasi-1D systems where VRH laws have been reported quite extensively, e.g. in polymers [22,23].

Equation (1) corresponds to *linear* variable-range hopping laws such that the current j is proportional to the applied electric field $\mathcal{E}$, $j = \sigma(T)\mathcal{E}$. Upon increasing the applied electric field, such laws cross-over to *non-linear* VRH laws which read:

$$j(\mathcal{E}) \sim \exp[-(\mathcal{E}_0/\mathcal{E})^{\gamma}], \tag{2}$$

where the exponent $\gamma \leq 1$. In the frame of disordered semiconductors in high electric fields such laws, with an exponent $\gamma = 1/4$, are known from Mott and Shklovskii. In quasi-1D systems they are observed, e.g. for bronzes [24]. These laws hold in the low-T limit, $T \ll 1/\tau_{el}$, up to the threshold field for global sliding of the pinned electronic structure, see references [25] and [26] for reviews on pinning. Of course, for the exponential character to manifest the electric field should not be too high, e.g. in 1D: $\mathcal{E} \ll \mathcal{E}_0 \propto 1/ev_F\tau_{el}^2$. For a given linear VRH law, it will however manifest for: $\mathcal{E} > \mathcal{E}_c$ where the crossover field reads, in any dimension: $\mathcal{E}_c = T\mathcal{E}_0/T_0 \propto T/e\xi$, where e is the unit charge and ξ is the localization length. In summary, we shall focus on the following two regimes: the linear response regime of equation (1):

$$T \ll T_0 \approx 1/\tau_{el} \text{ and } \mathcal{E} \ll T/e\xi, \tag{3}$$

and the non-linear response regime of equation (2):

$$T \ll T_0 \approx 1/\tau_{el} \text{ and } T/e\xi \ll \mathcal{E} \ll \mathcal{E}_0. \tag{4}$$

Some of the recent theoretical literature on VRH has been devoted to 1D and 3D quasi-1D systems in the linear regime, cf. reference [14]. To the knowledge of the author a theory of electron transport in 2D quasi-1D systems has not been developed in either the linear or non-linear regimes. Our specific task will be to compute the main exponential dependence of the electron current as a function of temperature, electric field and impurity concentration. We will closely follow the arguments of reference [14].

Our results display two characteristic features: *a non-monotonous dependence of the current as a function of disorder* and *a highly non-universal exponent* γ, i.e. implicitly interaction- and disorder-dependent. Both statements will be proved in the following sections. They are generic of *incommensurate quasi-1D Mott-Anderson insulators* and are absent in the usual doped semiconductors or even in the other (hypothetical) collective structure of interest: the pinned Wigner crystal. The reader interested more in our results than the details of the derivations may refer directly to Section 7.

The theory will be developed as follows. In Section 2 we describe the $4k_F$ quasi-1D systems in a semi-classical way, which is justified due to the large parameter r_s. This will unable us to determine the dielectric properties of the system. With the help of these results and the ES phenomenology we derive, in Section 3, the single-particle density of states (DOS) of charge excitations and show how it is affected by the long-range interaction potential among charge excitations (Coulomb gap). In Section 4 we focus on determining the localization length with the help of microscopic models related to single-impurity tunneling of the charge excitations. In Sections 5 and 6 the VRH laws, in the linear and non-linear response regimes, respectively, are derived with the help of the Efros, Mott and Shklovskii phenomenology and the results of Sections 3 and 4. In Section 7 the conclusion is given.

2 Dielectric properties

In this section we determine the dielectric properties of disordered 2D quasi-1D systems. The latter will be used in subsequent sections to determine the VRH laws.

2.1 The model

Following the Introduction, we introduce our basic model of a strongly pinned 2D quasi-1D system in a $4k_F-$CDW state. Along each wire, such a quasi-1D system is characterized by a modulation of the density:

$$\rho(x) = \rho_0 + \rho_1 \cos(Qx + \varphi) + (1/2\pi)\partial_x \varphi,$$

where ρ_0 is the unperturbed density, $Q = 4k_F$ is the modulation of the wave, φ is the phase of the (assumed incommensurate) CDW and the last term describes long-distance deformations. In electronic systems, a $4k_F$ modulation implies that the wavelength of the CDW is of the order of the average spacing a between electrons. It is realized in systems where the long-range Coulomb interaction is present, see reference [11] for a review on such results in the 1D case. In quasi-1D systems, where the Coulomb interaction may be screened by neighboring chains, the $4k_F-$CDW is also detected [17]. This reflects the large r_s nature of such systems. It implies that one may neglect quantum fluctuations and describe these CDWs semi-classically with the help of the phase-field φ. This CDW phase, φ, is related to the order-parameter describing the condensate which reads: $\Delta(\mathbf{r}) = |\Delta| \exp[i\varphi(\mathbf{r})]$, where $\mathbf{r}$ is a two-dimensional coordinate, and $|\Delta|$ the amplitude which is assumed to be frozen as we consider low temperatures. Moreover, the system has charge invariance, i.e. $\varphi \to \varphi + 2\pi$, implying the existence of a Fröhlich mode, i.e. the sliding of the electron crystal, in accordance with the fact that the system is incommensurate.

We focus first, in the frame of this semi-classical theory, on static properties of the system: the determination of the structure of the electron system in the presence of a single strong impurity and of it's charge excitations as well as the electrostatic potential between such excitations [the dynamics will be considered later, i.e. the effect of quantum fluctuations, when dealing with the tunneling of these charge carriers through the impurities]. Adding then a single strong impurity at the origin of the 2D system, the Hamiltonian consists of three parts:

$$H = H_0 + H_D + H_I, \tag{5}$$

where H_0 is the elastic part and reads:

$$H_0 = \int d^2\mathbf{r}\ \frac{Y}{2}\ \left[(\partial_x \varphi)^2 + \alpha\, (\partial_y \varphi)^2 \right], \tag{6}$$

where $Y = Y_x$ is the bulk-modulus along the chains and the anisotropy parameter reads: $\alpha = Y_y/Y_x \ll 1$ ($x-$ is the direction along the wires and $y-$ is the direction perpendicular to them). In equation (6), the phase field φ may be interpreted as a scalar displacement field along the wires, u_x, as in conventional elasticity theory. The displacement and phase fields are then related by: $\varphi = -(2\pi/a)u_x$. It follows naturally that the first term, in equation (6), describes the compression energy along each wire, Y being the bulk modulus and the second term corresponds to shear elasticity due to inter-wire interactions, Y_y being the shear modulus. It should be noticed that the inter-wire interaction term is defined at the level of the order parameter with the help of: $\Delta_n^* \Delta_m + \text{c.c.} = |\Delta|^2 \cos(\varphi_n - \varphi_m)$, where n and m index neighboring wires. Shear elasticity is then derived by expanding the cosine, at low temperatures, and going to the continuum limit perpendicular to the wires.

The second term in equation (5) describes the effect of a strong impurity at the origin and reads:

$$H_D = \int d^2\mathbf{r}\ [W_f \partial_x \varphi \delta(\mathbf{r}) - W_b \cos(\varphi)\delta(\mathbf{r})], \tag{7}$$

where the first term corresponds to the forward scattering on the impurity (i.e. the coupling of the point-like impurity potential to the long-distance part, $\propto \partial_x \varphi$, of the CDW density) and has a strength W_f. The second term corresponds to the backscattering on the impurity (i.e. the coupling of the point-like impurity potential to the oscillating, $4k_F$, part of the CDW density) with a strength W_b.

Finally, the third term in equation (5) contains the long-range Coulomb field and reads:

$$H_I = \int d^2\mathbf{r}\ \left[U\delta(\mathbf{r}) + \frac{1}{b^2}\partial_x \varphi U(\mathbf{r}) \right] - \frac{1}{8\pi e^2} \int d^3\mathbf{r} (\nabla U)^2], \tag{8}$$

where U is the Coulomb field. The first term corresponds to a point-like test charge for the Coulomb field U. The second to the coupling of the long-range Coulomb potential to the long-distance part of the density ($\propto \partial_x \varphi$) and the last term corresponds to the energy of the Coulomb field. It should also be noticed that, in the model of equation (8), the Coulomb interaction is the real one, i.e. the three-dimensional one.

There is a basic non-trivial length-scale in equation (6) which is proper to quasi-1D systems and will bring significant opportunities to go beyond strictly 1D physics all along this manuscript. This is the length of a $2\pi-$soliton: l_s, and emerges due to the fact that in quasi-1D systems shear and elasticity are coupled [27]. This can be understood with the help of the following arguments. An impurity will enforce a deformation of the CDW in its vicinity. In the absence of long-range Coulomb interaction, a deformation $\delta\varphi$ along a distance δx along the defected wire, has an elastic energy: $E = \delta x\, b^2[Y\delta\varphi^2/\delta x^2 + Y_\perp \delta\varphi^2/b^2]$, where we have assumed that the distance between neighboring wires is b. Minimizing this elastic energy with respect to δx and defining the optimal δx as the length l_s, we find that:

$$l_s = \frac{b}{\sqrt{\alpha}}. \tag{9}$$

In the case where $\delta\varphi = 2\pi$, the deformation becomes the plastic one, as a whole period of the density-wave is affected by the impurity. Hence the name of *soliton* length, see references [28] and [29] for reviews on soliton physics in condensed matter. Notice also that for decoupled wires, i.e. strictly 1D systems where $\alpha \to 0$, l_s is infinite. This implies that the whole wire adjusts to the deformation at the origin. In the case of $0 < \alpha < 1$, inter-wire interactions enforce the same phase $[2\pi]$ between neighboring wires, *beyond* the length l_s, on both sides of the impurity. In what follows, the length l_s will be used to distinguish between the low-impurity concentration limit, where the average distance between impurities along a wire, l, is larger than l_s, $l \gg l_s$ and the opposite large-impurity concentration limit, $l \ll l_s$ (the density of impurities is given by: $N = 1/lb^2$).

2.2 The case of small impurity concentration

We focus on the low-impurity concentration case, $l = 1/Nb^2 \gg l_s$ (recall that l is the average distance between impurities along a wire), cf. the one-impurity model of equation (5):

$$N \ll N_s = \frac{1}{l_s b} = \frac{\sqrt{\alpha}}{b^2}. \tag{10}$$

Each impurity is strong and the backscattering term in the Hamiltonian describes the pinning of the system at distances of the order of the average distance between impurities, l. On the other hand the forward scattering term describes softer deformations of the system away from the impurities [cf. the dipole distortions that we will introduce below]. We focus on the latter [30] and set: $W_b = 0$, assuming that the system is strongly pinned. By varying the functional of equation (5) with respect to the phase field and the Coulomb field, $\delta H/\delta\varphi = 0$ and $\delta H/\delta U = 0$ and going to Fourier space, one gets a system of two Poisson equations where both fields screen each other:

$$\varphi(\mathbf{q}) = \frac{iq_x}{\tilde{q}^2\varepsilon_2(\mathbf{q})}\left[\frac{W_f}{Y} + \frac{b}{r_D q}\right], \tag{11a}$$

$$U(\mathbf{q}) = \frac{2\pi e^2}{q\varepsilon_2(\mathbf{q})}\left[1 - \frac{W_f}{Yb}\frac{q_x^2}{\tilde{q}^2}\right], \tag{11b}$$

where $\tilde{q}^2 = q_x^2 + \alpha q_y^2$, $q^2 = q_x^2 + q_y^2$ and $r_D^{-1} = 2\pi e^2/Yb^2 \approx b^{-1}$ is the inverse screening length in the metallic phase. The 2D dielectric constant with the help of which the long-range Coulomb and elastic fields screen each-other reads:

$$\epsilon_2(\mathbf{q}) = \kappa\left[1 + \frac{q_x^2}{r_D q\tilde{q}^2}\right], \tag{12}$$

which displays the anisotropic metallic-like (in the sense of sliding of the collective electron structure) screening in the quasi-1D system. As can be seen from equations (11a) and (12), in 2D, the effect of the Coulomb interaction is to shift $\tilde{q}^2$ to $\tilde{q}^2\varepsilon_2(\mathbf{q}) = \tilde{q}^2 + q_x^2/r_D q$. This shift can be included in the bulk modulus and corresponds to the incompressibility of the crystal as a whole. As we are interested in physical properties at large distances along the chains $q_x \ll \sqrt{\alpha}q_y$ or $x \gg y/\sqrt{\alpha}$, we see that the Coulomb interaction brings into play the following sector:

$$q_x^2 \ll \alpha r_D q_y^3 \qquad y \ll (\alpha r_D x^2)^{1/3}, \tag{13}$$

which plays a dominant role as we show now.

The inverse Fourier transform of equation (11) yields:

$$\varphi(\mathbf{r}) = \frac{-b\,\mathrm{sgn}(x)}{6\pi(\sqrt{\alpha r_D}|x|)^{2/3}}\left[3\Gamma(\frac{5}{3}) - 2\Gamma(\frac{2}{3}, \frac{\sqrt{\alpha r_D}|x|}{y^{3/2}})\right], \tag{14a}$$

$$U(\mathbf{r}) = \frac{e^2}{\kappa|x|}\left[1 - \exp\left(-\frac{\sqrt{\alpha r_D}|x|}{y^{3/2}}\right)\right], \tag{14b}$$

where $\Gamma(x)$ and $\Gamma(x,y)$ are the complete and incomplete gamma functions, respectively. The phase field of equation (14a) is an odd function of the coordinate along the wire and at long distances along the wire, i.e. for $y \ll (\alpha r_D x^2)^{1/3}$, reads: $\varphi(\mathbf{r}) \propto -(l_s/x)^{2/3}$, cf. equation (15a) below. This implies that, at such distances, the impurity is surrounded by *dipole distortions* of length l_s. Crucially, we identify these dipole distortions as the charge excitations of the system.

These charge excitations interact with a Coulomb potential which deviates from the usual 3D one, due to the anisotropic screening, cf. equation (14b). More generally, equation (11b) shows that the electrostatic potential changes sign along the cone: $y = \pm\sqrt{\alpha}|x|/(W_f/Yb - 1)$. This is related to the dipole nature of the phase deformations. In particular, outside this cone, the potential reads: $U \propto -W_f r_D/r^3$, so that charge deformations of the same sign attract each-other via a dipole potential. On the other hand, within this cone (which is the sector of validity of equation (14b)) the potential is repulsive and independent of W_f. This implies that, if they were not bound to the impurities (which they actually originate from), such charge excitations would form domain walls (our arguments have their roots in Ref. [32]). In the following we return on the sector within the cone: $y < \pm\sqrt{\alpha}|x|/(W_f/Yb - 1)$, where the potential is repulsive and equations (14) holds.

We summarize this sub-section by giving the asymptotic expressions of equations (14), on one hand close to the chains, i.e. in the sector of equation (13), $y \ll (\alpha r_D x^2)^{1/3}$, where it reads:

$$\varphi(\mathbf{r}) = -\mathrm{sgn}(x)\ \left(\frac{l_s}{2\pi|x|}\right)^{2/3}, \tag{15a}$$

$$U(\mathbf{r}) = \frac{e^2}{\kappa|x|}, \tag{15b}$$

and on the other hand further away from the chains, i.e. $y \gg (\alpha r_D x^2)^{1/3}$, where it reads:

$$\varphi(\mathbf{r}) = -\frac{b\ \mathrm{sgn}(x)}{2\pi|y|}, \tag{16a}$$

$$U(\mathbf{r}) = \frac{e^2\sqrt{\alpha r_D}}{\kappa y^{3/2}}. \tag{16b}$$

In the limit of vanishing inter-wire coupling: $\alpha \to 0$, the sector defined by equation (13) vanishes. This corresponds effectively to a crossover from a low impurity density regime, $N \ll N_s = 1/l_s b^2 = \sqrt{\alpha}/b^3$, to a large impurity density regime, $N \gg N_s$, that we consider next.

2.3 The case of large impurity concentration

Formally, this case, as defined in the previous section, requires that $l \ll l_s$ (recall that l is the average distance between impurities along a wire) or, in terms of impurity concentration, that:

$$N \gg N_s = \frac{\sqrt{\alpha}}{b^2}, \tag{17}$$

where α gives the dimensionless strength of inter-chain interactions. Large impurity concentration is therefore equivalent to vanishing inter-chain couplings. The system is then equivalent to an ensemble of metallic (in the sense of sliding) segments along the wires, of average length l. These segments are decoupled elastically but still coupled by the long-range Coulomb potential. In the literature, such a regime is sometimes referred to as a model of interrupted metallic strands [31]. Following the previous paragraph, our goal here is to determine the dielectric properties of such a phase.

From equations (11), the electrostatic potential of a charge carrier at the origin of a *pure* system ($W_f = W_b = 0$) reads:

$$U(\mathbf{q}) = \frac{2\pi e^2}{q\epsilon_2(\mathbf{q})}, \tag{18}$$

where the dielectric constant is given by equation (12) and $q^2 = q_x^2 + q_y^2$. In the previous paragraph we have focused on the effect of deformations due to the forward scattering term (W_f) in a system of dilute strong pinning centers. Increasing the impurity concentration we need to take into account the average effect of these pinning center. This is done according to the prescription: $\tilde{q}^2 \to \tilde{q}^2 + L_x^{-2}$ in equation (12) for the dielectric constant (recall that $\tilde{q}^2 = q_x^2 + \alpha q_y^2$ arises from the elastic part of the energy) where $L_x \propto l$ is the pinning length due to the backscattering on the impurity, see references [25] and [26] for reviews on pinning. This amounts to introduce the effect of backscattering on the impurity as a commensurability term explicitly breaking the translational invariance of the system. The dielectric constant therefore becomes:

$$\epsilon_2(\mathbf{q}) = \kappa\left[1 + \frac{q_x^2}{r_D q(\tilde{q}^2 + L_x^{-2})}\right]. \tag{19}$$

In the clean limit, $L_x \propto l \gg l_s$, we may assume that $L_x/l_s \to \infty$, and recover the results of the previous subsection. In the dirty limit, $L_x \propto l \ll l_s$, we may assume that $l_s/L_x \to \infty$ so that $\alpha \to 0$ and $\tilde{q}^2 = q_x^2 + \alpha q_y^2 \to q_x^2$. In this regime, the Coulomb potential reads:

$$U(\mathbf{r}) = \frac{2\pi e^2}{\kappa}\int\frac{d\mathbf{q}}{(2\pi)^2}\ \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q + \frac{q_x^2}{r_D\ (q_x^2 + L_x^{-2})}}. \tag{20}$$

The asymptotics of equation (20) then read:

$$U(\mathbf{r}) = \frac{e^2}{\kappa|x|}\left(1 - \exp\left(-\frac{|x|}{\sqrt{yb\kappa_x/\kappa}}\right)\right),$$
$$L_x \ll x \ll L_x^2/2b, \quad y \ll L_x^2/2b, \tag{21a}$$

$$U(\mathbf{r}) = \frac{e^2}{\kappa y}, \quad L_x \ll x \ll L_x^2/2b, \quad y \gg L_x^2/2b, \tag{21b}$$

$$U(\mathbf{r}) = \frac{e^2}{\kappa r}, \quad r \gg L_x^2/2b, \tag{21c}$$

where $r = \sqrt{x^2 + y^2}$ and κ_x is the longitudinal (along the wires) dielectric constant:

$$\kappa_x = \kappa(L_x/r_D)^2, \tag{22}$$

up to a numerical constant of the order of unity, with: $r_D \approx b$, where b is the inter-wire distance and the pinning length along the wires: $L_x \approx l$, is of the order of the average impurity-distance, l, along each wire. Notice that the longitudinal dielectric constant, equation (22), is inversely proportional to the square of the impurity concentration in the system, $N = 1/lb$. This longitudinal dielectric constant is large, i.e. larger than the dielectric constant κ of the host in which the 2D system is embedded. This is important and explains the richness of equations (21). Indeed, equation (21a) shows that close to the chains ($y \ll l^2/2b$) and for $l \ll x \ll l^2/2b$, field lines prefer to remain in the 2D layer where they are screened by the large longitudinal dielectric constant. This sector is peculiar to the quasi-1D system. At larger distances, $x \gg l^2/2b$, the lines escape to the external media where they are screened only by κ and the potential is fully long-ranged. This result is derived by other means in the appendix.

3 Single-particle density-of-states and the Coulomb gap

We now turn to the determination of the single-particle density of states, $g(\varepsilon)$, of the localized charge excitation of energy ε. This density of states is defined as the probability density for the excitation to have energy ε. In the case where $\varepsilon = \varepsilon_i$, the corresponding wave function is given by:

$$\psi_i(\mathbf{r}) = \exp\left[-\frac{x_i}{\xi_x} - \frac{y_i}{\xi_y}\right], \tag{23}$$

where ξ_x is the longitudinal, i.e. along the wires, localization length and ξ_y is the transverse one. Moreover, x_i and y_i locate the position of the localized charge excitation of energy ε_i. Notice that, in usual disordered semiconductors, ψ is the single-particle wave function of the electron (impurities are hydrogenoid atoms) and the localization length corresponds to the Bohr radius of this trivial charge excitation. In quasi-1D systems ψ is the single-particle wave function of deformations of the electronic system, the distortions considered in the previous section.

An ensemble of such localized states with a density $g(\varepsilon)$, forms a disorder or impurity band. Moreover, keeping in mind that the system is a Mott insulator, the localized states of energy ε belong the lowest Hubbard band of the system. It is on this lowest "Hubbard impurity-band" that we shall focus now, keeping in mind that the upper one, distant by the Hubbard: $U_H \sim e^2/\kappa a$, is unreachable at the low energies we consider: $T \ll U_H$.

We further assume that *this lowest Hubbard impurity band is partially filled*. As a result, the low-T transport we consider is the hopping of charge excitations within this lowest Hubbard impurity band. One may formally define a Fermi energy for this band of localized states ($\varepsilon_F \equiv 0$) and a non-zero DOS of charge excitation at this Fermi level: $\nu \equiv g(0) \neq 0$. As far as transport is concerned, we shall show below that the non-zero ν, i.e. the existence of gapless charge excitations, implies that there is a low-temperature hopping conductivity of the variable-range type.

In the absence of the long-range Coulomb $g(\varepsilon)$ is constant, i.e. $g(\varepsilon) \approx \nu$, within the disorder band width. We now turn on the long-range Coulomb interaction and follow Efros and Shklovskii to determine the influence of this interaction on $g(\varepsilon)$. We provide here some details for the reader which is not familiar with the ES arguments. While hopping from the localized state i of energy ε_i below the effective Fermi energy of the impurity-band to the state j of energy ε_j above, the energy of the charge excitation will vary by: $\delta E = \varepsilon_j - e^2/\kappa r_{ij} - \varepsilon_i$. In the latter expression, the long-range Coulomb interaction $-e^2/\kappa r_{ij}$ between the charge excitation at j and the hole it has left at i has been taken into account. By construction: $\delta E \geq 0$, which implies a *depletion* of states around the Fermi energy: $r_{ij} \geq e^2/\kappa|\varepsilon_j - \varepsilon_i|$, because of the long-range nature of the Coulomb interaction. In d dimensions, such states have a spatial density of $n = 1/r^d$, where $r \equiv r_{ij}$. This implies that: $n(\varepsilon) \leq (\kappa|\varepsilon|/e^2)^d$, where $\varepsilon \equiv \varepsilon_j - \varepsilon_i$. The corresponding single-particle density of states ($g(\varepsilon) = dn(\varepsilon)/d\varepsilon$) then reads:

$$g(\varepsilon) = C\,\left(\frac{\kappa}{e^2}\right)^d\,|\varepsilon|^{d-1}, \tag{24}$$

where C is a numerical coefficient of the order of unity, depending on the dimensionality of the system [10], e.g. see reference [33] for the 2D case we are interested in. A crucial feature of equation (24) is that the DOS vanishes only at the Fermi energy. Hence, there is still no Coulomb blockade within the impurity band because of the long-range Coulomb. There is however a soft Coulomb-gap which affects the transport as will be shown below. Moreover, the Coulomb gap of equation (24) depends on dimensionality [10]: quadratic in the energy of the charge excitation in 3D ($\propto \varepsilon^2$) and linear in 2D ($\propto |\varepsilon|$), and not on the impurity concentration. The situation is much richer in quasi-1D systems due to their non-trivial dielectric properties.

3.1 The case of large impurity concentration

We apply the Efros-Shklovskii arguments to the case of the anisotropic Coulomb interaction characteristic of quasi-1D systems starting with the large-impurity concentration regime: $N \gg N_s$. The Coulomb interaction is given by equations (21). For each sector we fix the potential $U = \varepsilon$, determine the equipotentials $x(\varepsilon)$ and $y(\varepsilon)$ and substitute them in the density, $n(\varepsilon)$ in order to derive the Coulomb gap:

$$n(\varepsilon) = \frac{1}{x(\varepsilon)y(\varepsilon)}, \quad g(\varepsilon) = \frac{dn(\varepsilon)}{d\varepsilon}.$$

We first focus on the large-distance sector of equation (21c). At such distances, $x \gg l^2/2b$, the potential is isotropic. Therefore the usual ES arguments apply, $x(\varepsilon) = y(\varepsilon) = e^2/\kappa\varepsilon$ and $n(\varepsilon) = \kappa^2\varepsilon^2/e^4$. The Coulomb gap is therefore linear in the energy of the charge excitation:

$$g_{ES}(\varepsilon) = C_0\,\frac{\kappa^2}{e^4}\,|\varepsilon|, \quad \varepsilon \ll \varepsilon_1, \tag{25}$$

where C_0 is a numerical coefficient of the order of unity, ε_1 is a crossover to another Coulomb-gap shape (as explained shortly). As in the usual ES law, equation (25) is independent on the impurity concentration (C_0 and κ are constants).

We then focus on the shorter-distance sector of equation (21a). At such distances, $l \ll x \ll l^2/2b$ and $y \ll l^2/2b$ (recall that $L_x \propto l$), the potential reads:

$$U(\mathbf{r}) = \frac{e^2}{\kappa x}, \qquad y \ll \frac{x^2\kappa}{b\kappa_x}, \tag{26a}$$

$$U(\mathbf{r}) = \frac{e^2}{\kappa\sqrt{yb\kappa_x/\kappa}}, \quad y \gg \frac{x^2\kappa}{b\kappa_x}. \tag{26b}$$

which follows from equation (21a). Close to the chains, $y \ll x^2\kappa/b\kappa_x$, and fixing $U \equiv \varepsilon$ yields the following equipotentials: $x(\varepsilon) = e^2/\kappa\varepsilon$ and $y(\varepsilon) = x(\varepsilon)^2\kappa/a\kappa_x$. This yields a density of localized states:

$$n(\varepsilon) = \frac{b\kappa^2\kappa_x\varepsilon^3}{e^6},$$

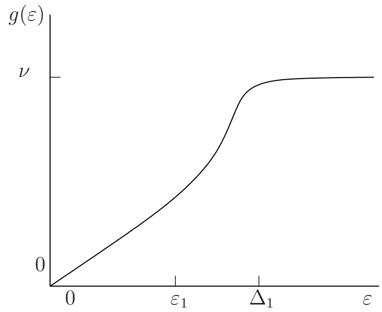

Fig. 2. Schematic plot of the DOS of charge excitations of the 2D anisotropic systemwith a large concentration of impurities, $N \gg N_s$. The linear Coulomb-gap at low energies (Eq. (25)) is succeeded by the quadratic Coulomb-gap (Eq. (27)) and finally by the constant DOS of the disorder band, ν.

and therefore the Coulomb gap:

$$g_1(\varepsilon) = C_1 \, \frac{b\kappa^2\kappa_x\varepsilon^2}{e^6}, \quad \varepsilon_1 \ll \varepsilon \ll \varDelta_1, \tag{27}$$

where C_1 is a numerical coefficient of the order of unity. This Coulomb gap is *quadratic* for a 2D system and depends on disorder through κ_x, cf. equation (22). Both features are unusual with respect to known results, cf. equation (24). In equation (27), $\varDelta_1$ is the width of the Coulomb gap and ε_1 a crossover energy from equation (25) to equation (27). The upper-bound $\varDelta_1$ to the energy-dependence of equation (27) originates from the fact that this Coulomb gap is due to the short-distance part of the Coulomb potential equation (21a). At shorter-distances, hence higher energies, equation (27) crosses-over to the constant DOS, ν, of the disorder band. Equating $g_1(\varDelta_1)$ to ν yields:

$$\varDelta_1 = D_1 \, \left[e^6\nu/b\kappa^2\kappa_x\right]^{1/2}, \tag{28}$$

where D_1 is a numerical coefficient of the order of unity. equation (28) corresponds to the Coulomb-gap width in the large impurity regime. On the other hand, the crossover energy ε_1 is obtained by equating $g_{ES}(\varepsilon_1)$ to $g_1(\varepsilon_1)$ which, from equations (25) and (27), yields:

$$\varepsilon_1 = N_1 \, \frac{e^2}{b\kappa_x}, \tag{29}$$

where N_1 is a numerical coefficient of the order of unity. Notice that both equations (28) and (29) have an impurity-dependence through the longitudinal dielectric constant κ_x which is given by equation (22). In the large impurity concentration case, the total Coulomb gap shape is plotted in Figure 2.

3.2 The case of small impurity concentration

We turn on to the small-impurity concentration case: $N \ll N_s$. The ES Coulomb gaps found in the case of large impurity concentration, $N \gg N_s$, are still valid in the present case. Hence, at distances x larger than $l^2/2b$, interactions open a linear Coulomb gap similar to the one of equation (25). At smaller distances: $l \ll x \ll l^2/2b$, corresponding to larger energies, this linear Coulomb gap crosses over, at ε_1, to the quadratic one defined by equation (27). Going to higher energies corresponds to distances shorter than the pinning length, $L_x \propto l$. If impurities are sufficiently diluted, i.e. l is large, we may face the situation where a charge excitation experiences the potential within the pinning area $L_xL_y = \sqrt{\alpha}l^2$, i.e. the potential of the equivalent pure system given by equations (15b) and (16b):

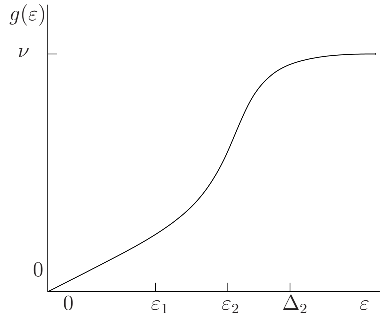

Fig. 3. Schematic plot of the DOS of the 2D anisotropic system with a small concentration of impurities, $N \ll N_s$. The linear Coulomb-gap at low energies (Eq. (25)) is succeeded by the quadratic Coulomb-gap (Eq. (27)), then by the "2/3-"Coulomb-gap (Eq. (31)) and finally by the constant DOS of the disorder band, ν.

$$U(\mathbf{r}) = \frac{e^2}{\kappa|x|}, \qquad y \ll [x^2\alpha r_D]^{1/3} \tag{30a}$$

$$U(\mathbf{r}) = \frac{e^2}{\kappa}\left[\frac{\alpha r_D}{|y|^3}\right]^{1/2}, \quad y \gg [x^2\alpha r_D]^{1/3}. \tag{30b}$$

Close to the chains, $y \ll [x^2\alpha r_D]^{1/3}$, and fixing $U \equiv \varepsilon$ yields the following equipotentials: $x(\varepsilon) = e^2/\kappa\varepsilon$ and $y(\varepsilon) = [x^2(\varepsilon)\alpha r_D]^{1/3}$. This yields a density of localized states:

$$n(\varepsilon) = \frac{1}{(\alpha r_D)^{1/3}} \, \left(\frac{\kappa\varepsilon}{e^2}\right)^{5/3},$$

and therefore the Coulomb gap:

$$g_2(\varepsilon) = C_2 \, \frac{\varepsilon^{2/3}}{(\alpha r_D)^{1/3}(e^2/\kappa)^{5/3}}, \quad \varepsilon_2 \ll \varepsilon \ll \varDelta_2, \tag{31}$$

where C_2 is a numerical coefficient of the order of unity. In equation (31), the new crossover energy, ε_2, defined as the crossover energy between the quadratic and $2/3-$Coulomb gaps $[g_1(\varepsilon_2) = g_2(\varepsilon_2)]$ reads:

$$\varepsilon_2 = N_2 \, \frac{e^2}{\kappa b} \, \left[\frac{1}{\alpha}\left(\frac{\kappa_x}{\kappa}\right)^3\right]^{1/4}, \tag{32}$$

where N_2 is a numerical coefficient of the order of unity. Finally, $\varDelta_2$ corresponds to the Coulomb-gap width in the small-impurity concentration case $[g_2(\varDelta_2) = \nu]$ and reads:

$$\varDelta_2 = D_2 \, \left[(e^2/\kappa)^5\nu^3\alpha r_D\right]^{1/2}, \tag{33}$$

where D_2 is a numerical coefficient of the order of unity. These results are summarized in Figure 3 which displays the total DOS of the system in the small impurity concentration regime.

4 Localization length and tunneling

We now focus on the localization length associated with the single-particle wave function of equation (23). The transverse localization length is given by: $\xi_y < b$, where b is the inter-chain distance. In the following we focus on ξ_x. As we have already said at the level of equation (23), in the case of doped semiconductors, the charge excitations are simply the electrons and the corresponding localization length is the Bohr radius of the hydrogenoid wave function. The later has no impurity dependence. On the other hand, the situation is much less trivial for quasi-1D systems, where excitations are non-linear deformations of the electronic system. These excitations are extended and one may think that they are localized over the length $\xi_x = l_s$ along the wires, in the clean limit ($l \gg l_s$) and over the distance $\xi_x = l$ between impurities in the dirty limit ($l \ll l_s$). This intuitive argument is correct but it turns out that the Föhlich mode as well as the long-range Coulomb interaction also influence, in a non-trivial way, this length, as we show now. The reader which does not wish to go through our microscopic arguments, at least on the first reading, may skip to the next section, where transport laws are derived and it is shown how the localization lengths enter expressions for the conductivity and current.

4.1 The model

It is crucial to notice that the localization length is of purely quantum origin, contrary to the pinning length (Fukuyama-Lee length or Larkin length depending on the context) which is defined at the classical level. The derivation of the localization length requires to consider the tunneling of the charge excitations through impurities. The most convenient way of dealing which such processes, which are forbidden at the classical level, is to extend the semi-classical approach of previous sections to the Euclidean space by using imaginary time dynamics. In doing so, we consider the tunneling of a charge excitation along the distance x of a given wire. The average distance between impurities along this wire is l. Hence, the charge has to tunnel through x/l impurities. Neglecting interferences between these impurities, their effect, as a first approximation, is additive. The total action for tunneling along the distance x is therefore given by: $S_{\rm opt}(x) = s_{\rm opt} x/l$, where $s_{\rm opt}$ is the action to tunnel through *a single* impurity. The total action $S_{\rm opt}$ enters the quantum probability to reach the distant point x: $|\psi(x)|^2 \sim \exp(-S_{\rm opt}(x))$, where ψ is the one-(quasi-)particle wave function of equation (23). Therefore: $x/\xi_x \equiv S_{\rm opt} = s_{\rm opt} x/l$ and the localization length reads:

$$\xi_x = l/s_{\rm opt}, \quad l \ll l_s. \tag{34}$$

This result is valid for the large impurity concentration case. When the concentration of impurity is small charge excitations acquire the length l_s and the localization length becomes:

$$\xi_x = l_s/s_{\rm opt}, \quad l \gg l_s. \tag{35}$$

In our approximation of neglecting interferences between various impurities, which is reasonable in the strong or individual pinning regime we consider, the localization length depends only on a one-impurity tunneling action. Subsequent calculations will therefore focus on a derivation of this one-impurity optimal action $s_{\rm opt}$.

The basic action that we shall consider reads:

$$s = \int_0^\beta d\tau \int d\mathbf{r} \left\{ \frac{C}{2}(\partial_\tau\varphi)^2 + \frac{Y}{2}\left[(\partial_x\varphi)^2 + \alpha\,(\partial_\perp\varphi)^2\right] - W_b\cos(\varphi)\delta(\mathbf{r}) \right\}, \tag{36}$$

where now $\varphi \equiv \varphi(\tau, \mathbf{r})$, C is the CDW stiffness, the last term corresponds to the backscattering on the impurity and $\hbar = 1$ unless specified explicitly. At this point we came up to a systematic way of deriving the localization length with the help of equations (34), (35) and the well-defined model of equation (36). Unfortunately, it is not possible to determine analytically the exact non-trivial time-dependent solutions (instantons) of the above non-linear partial differential equation. The physical process of tunneling may however be understood as a two-stage process [34]. The first stage (small times) corresponds to the local tunneling of the CDW at the impurity position by one period, a. This stage is described by the backscattering term which is local in space. We now assume that the non-trivial dynamics of the phase at the impurity are essential only at large times (large and small times will be defined below). As a result the jump of the phase, at the impurity, during the first stage is described by the Ansatz: $\varphi(\tau) = 2\pi\theta(\tau)$, where $\theta(\tau)$ is the Heaviside function. The action describing the first stage is then linear in time and in the impurity strength:

$$s_1 = W_b\beta. \tag{37}$$

The second stage (large times), corresponds to the adjustment of the crystal, at large distances from the impurity, to the jump of the phase at the impurity. It is described by assuming that the local effect of the impurity is irrelevant, i.e. that one can set $W_b = 0$. Then, the corresponding effective action may be derived from equation (36) (with $W_b = 0$) by tracing out all remaining gapless modes away from the impurity position. This leads to an effective action for the phase at the impurity:

$$s_2 = \frac{1}{2\beta}\sum_\omega |\varphi_\omega|^2 J_\omega, \tag{38}$$

where the spectral function J_ω contains the effect of the infinite number of gapless modes which have been traced out. Without the Coulomb interaction this spectral function reads:

$$J_\omega^{-1} = \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{C\omega^2 + Y\tilde{q}^2}. \tag{39}$$

where $\tilde{q}^2 = q_x^2 + \alpha q_y^2$.

4.2 The case of large impurity concentration

In this case wires are elastically weakly coupled, $\alpha \to 0$. This case is worth examining first because it relates our approach to known results in the literature on 1D disordered interacting systems. equation (39) leads to $J_\omega = 4\pi b\sqrt{YC}|\omega|$ and therefore to an action of equation (38) which is non-local in time:

$$s_2 = \int d\tau d\tau' \left(\frac{\varphi(\tau) - \varphi(\tau')}{\tau - \tau'} \right)^2,$$

and is similar to the one of Caldeira and Leggett in the frame of quantum dissipation. This terminology might be misleading here, as the dissipation arising in the effective action is not related to an external bath coupled to the system (phonons). It is intrinsically related to the incommensurate CDW, i.e. the action is that of generic gapless modes. It may be understood by thinking about the CDW away from the impurity as an effective (internal) bath governing the dynamics of the phase at the impurity. Tracing out the internal degrees of freedom away from the impurity, while keeping the phase at the impurity fixed, leads to non-trivial dynamics for the latter. This procedure is well-known in 1D disordered systems, cf. references [5,35,36], the first extension to quasi-1D systems appearing in reference [26]. Because the phase is bounded along the time trajectory, i.e. $\Delta\varphi = 2\pi$, this part of the action is logarithmic in time:

$$s_2 = b\,\sqrt{YC}\,\ \ln(l/u\beta), \tag{40}$$

where $u = \sqrt{Y/C}$ is the velocity of the collective electron structure and l/u an upper-time cut-off preventing the divergency of the action.

With the help of equations (37) and (40), minimizing the total action $s = s_1 + s_2$ with respect to β, the optimal time reads:

$$\beta_{\text{opt}} = b\,\sqrt{YC}/W_b, \tag{41}$$

and the optimal action reads:

$$s_{\text{opt}} = b\,\sqrt{YC}\,\ \ln(lW_b/bY). \tag{42}$$

In equation (42), we have assumed that the logarithm gives the major contribution with respect to unity. This is effectively the case, as the inequality: $W_b l/b \gg Y$, is equivalent to the requirement that we are in the strong pinning regime. The latter also implies that the impurity strength and bulk modulus are determined by local electrostatics: $W_b \sim e^2/\kappa b$ and $Y \sim e^2/\kappa b$, as may be check by dimensional arguments. The arguments of the logarithm therefore corresponds to: $W_b l/bY = l/b \gg 1$.

The second check deals with our initial assumption that at small times, the non-trivial dynamics of the phase at the impurity position are irrelevant. We may check that this is the case for an infinite, delta-function-like, back-scatterer ($W_b \to \infty$). In this case, the tunneling is instantaneous and the phase effectively jumps by 2π, i.e. by one period at the impurity position. The Ansatz: $\varphi(\tau) = 2\pi\theta(\tau)$, where $\theta(\tau)$ is the Heaviside function, is then perfectly justified. This is also the case if the optimal time is smaller or equal to the smallest time-scale of the problem: $\beta_{\text{opt}} \leq a/u$. Using again the fact that the plasmon velocity is related to our parameters with the help of: $u = \sqrt{Y/C}$, this condition reads: $W_b \geq Yb/a$. The lower boundary is satisfied in the case of strong individual pinning, $W_b \sim Y \sim e^2/\kappa b$, for sufficiently dilute electronic systems (hence large r_s, see below).

Finally, we notice that the phase-phase correlation function in the absence of disorder scales as: $<\varphi^2> = \hbar/b\sqrt{YC}$ (where $\hbar$ has been restored for clarity). The weakness of quantum fluctuations, on which our starting semi-classical approach was based, implies that: $b\sqrt{YC} \gg \hbar$. This condition fulfills the requirement that, in equation (42), the optimal action, $s_{\text{opt}} \gg 1$. Because $u = \sqrt{Y/C}$, and using again the fact that: $Y \sim e^2/\kappa b$, we see also that: $b\sqrt{YC}/\hbar = e^2/\kappa/\hbar u = U_H/\epsilon_F$, where $U_H = e^2/\kappa a$ is the Coulomb energy scale and $\epsilon_F = \hbar u/a$ is the kinetic energy scale (a is the average distance between electrons along a wire). We may therefore introduce the well-known parameter r_s which is defined as:

$$r_s \equiv \frac{U_H}{\epsilon_F} = b\sqrt{YC}/\hbar \gg 1. \tag{43}$$

It follows from equation (43), that the optimal action of equation (42), may be re-expressed as:

$$s_{\text{opt}} = r_s \ln(l/b), \tag{44}$$

where we have used the fact that $W_b \sim Y$ and returned $\hbar$ to unity. It is crucial to notice that this action is non-WKB like because the strength of the barrier, W_b, is in the logarithm, as known from Larkin and Lee in the strictly 1D case [34]. The difference with the 1D case here is that the inter-chain distance, b, appears in the logarithm instead of the average electron distance, a, in 1D. From equations (34) and (44), this yields the localization length in the large impurity concentration case:

$$\xi_x = \frac{1}{r_s bN \ln(1/Nb^2)}, \quad N \gg N_s, \tag{45}$$

where $N = 1/lb$ has been used.

We include now the long-range Coulomb. Following equation (12) and the discussion below it, the inclusion of the Coulomb interaction amounts to replace: $\tilde{q}^2 = q_x^2 + \alpha q_y^2 \to q_x^2$ by $\tilde{q}^2\varepsilon_2(\mathbf{q}) = \tilde{q}^2 + q_x^2/r_D q \to q_x^2[1 + 1/r_D q]$, where $q^2 = q_x^2 + q_y^2$. Because $qr_D \ll 1$ and at large distances along the chains, $q_x \ll q_y$, this reduces to the non-analytic shift: $\tilde{q}^2 \to q_x^2/r_D|q_y|$ in equation (39), which reads:

$$J_\omega^{-1} = \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{C\omega^2 + Yq_x^2/r_D|q_y|}.$$

It is straightforward to show that the integration yields: $J_\omega = 3\pi b\sqrt{YC}|\omega|$, which is the above result up to a numerical factor. In 2D and in the large impurity concentration case, the Coulomb interaction does not modify the result of equation (45).

The case where $N \ll N_s$ requires stronger inter-wire couplings and will be considered next.

4.3 The case of small impurity concentration

In the small-impurity concentration case the inter-wire coupling is crucial, i.e. it gives rise to the non-trivial scale l_s. For coupled wires, we are only aware of the results of reference [26] dealing with the equivalent 3D geometry. In our 2D case, including inter-chain interactions, and focusing on large-distances along the chains, $q_x \ll \sqrt{\alpha}q_y$, equation (39) reads:

$$J_\omega^{-1} = \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{C\omega^2 + \alpha Y q_y^2}.$$

This yields:

$$J_\omega = \frac{E_s}{\ln(E_s/b\sqrt{YC}|\omega|)}, \qquad (46)$$

where $E_s \propto e^2/\kappa l_s$ is the soliton energy. The total action, $s = s_1 + s_2$, in the logarithmic approximation, therefore reads:

$$s = W_b\beta + \frac{E_s\beta}{\ln(E_s\beta/b\sqrt{YC})}.$$

The logarithmic factor provides a minimum to the action which, in the strong pinning regime, $W_b \gg E_s$, is then given by:

$$s_{\mathrm{opt}} = b\sqrt{YC}\,\frac{W_b}{E_s} = r_s\,\frac{W_b}{E_s}, \qquad (47)$$

where equation (43) has been used.

Next, we include the effect of the Coulomb interaction shifting $\tilde{q}^2$ to $\tilde{q}^2\epsilon_2(\mathbf{q}) = \tilde{q}^2 + q_x^2/r_D q$, cf. equation (12) and the discussion following it. At large distances along the chains, $q_x \ll \sqrt{\alpha}q_y$, the shift reduces to $\tilde{q}^2 \to \alpha q_y^2 + q_x^2/r_D|q_y|$. The kernel of equation (39) then reads:

$$J_\omega^{-1} = \int \frac{d\mathbf{q}}{(2\pi)^2} \frac{1}{C\omega^2 + \alpha Y q_y^2 + Y q_x^2/r_D|q_y|}.$$

The integrations yield:

$$J_\omega = \frac{E_s}{1 - (b\sqrt{YC}|\omega|/E_s)^{1/2}}. \qquad (48)$$

equation (48) shows that, in the presence of the Coulomb interaction, the dynamics enter the kernel in a perturbative way with respect to the static part E_s. This was not the case in 1D as well as in 2D without the long-range Coulomb interaction, cf. the logarithmic factor in equation (46). The expansion of the kernel, up to second order, leads to the following total action:

$$s = W_b\beta + [bE_s\sqrt{YC}\beta]^{1/2} + b\sqrt{YC}\,\ln(l_s/u\beta),$$

where our cut-off is now l_s. The optimal time is found to be:

$$\beta_{\mathrm{opt}} = \frac{b\sqrt{YC}}{W_b}. \qquad (49)$$

which leads to the following optimal action:

$$s_{\mathrm{opt}} = b\sqrt{YC} + b\sqrt{YC}\left(\frac{E_s}{W_b}\right)^{1/2} + b\sqrt{YC}\,\ln\left(\frac{l_s W_b}{bY}\right).$$

In the strong pinning regime, $W_b \sim Y$, and with $l_s \gg b$, the action is dominated by the logarithmic contribution which, up to a numerical factor, reads:

$$s_{\mathrm{opt}} = r_s\,\ln(l_s/b). \qquad (50)$$

where equation (43) has been used as well as the strong pinning result: $W_b \sim Y$. The long-range Coulomb therefore returns us to the logarithmic action which is non WKB-like.

From equations (35) and (50), the localization length in the small impurity concentration case reads:

$$\xi_x = \frac{1}{r_s b N_s \ln(1/N_s b^2)}, \quad N \ll N_s, \qquad (51)$$

where $N = 1/lb$ has been used ($N_s = 1/l_s b$).

4.3.1 General expression and remark

Equations (45) and (51) yield the general expression for the localization length:

$$\xi_x^{-1} = r_s\, b\, \max\{N, N_s\}\,\ln\left(\frac{1}{\max\{N, N_s\}b^2}\right), \qquad (52)$$

which we will use in the following section.

As a final note, we follow reference [26], where it has been mentioned that the collective dynamics of the CDW may include a contribution from the amplitude, $|\Delta(\tau)|$. This contribution arises because the order parameter is complex: $\Delta(\mathbf{r}, t) = |\Delta|\exp(i\varphi)$. At zero temperature the amplitude mode is frozen but in VRH we may be interested in reaching the thermally activated regime. In this case, there is an additional regular kinetic energy contribution $\propto I\omega^2$, where the momentum of inertia I depends on fluctuations of the amplitude. This brings an additional contribution to the total action of equation (50): $\propto I/\beta$. Equation (50) would then be valid provided that: $I \ll b^2YC/W_b$. In the other case, $I \gg b^2YC/W_b$, the final action reads:

$$s_{\mathrm{opt}} = 2\sqrt{IW_b} + b\sqrt{YC}\,\ln\left(\frac{L\sqrt{W_b}}{u\sqrt{I}}\right), \qquad (53)$$

where the second term is a correction and $L = \min\{l, l_s\}$. Notice that the first term, with a square-root dependence on the barrier, is WKB-like. In the following, we will assume that the amplitude mode has a negligeable contribution and we will use the final expression for the localization length, equation (52), based on the non WKB-like actions. Depending on the system under consideration the WKB contribution may however play a significant role. One would then have to use equation (53) together with equations (34) and (35) to derive the localization length.

5 Linear VRH laws

This section deals with the hopping laws of the quasi-1D Mott-Anderson insulators in the linear response regime. Our results depend crucially on the Coulomb gap shapes derived in Section 3 and on the impurity-dependence of the longitudinal localization length derived in Section 4. Notice that all transport laws below, and especially the parameter characteristic of these laws, are given up to a numerical coefficient of the order of unity.

Recall that the semi-phenomenological arguments initially introduced by Mott in order to derive the dc conductivity, $\sigma(T)$, are based on minimizing (with respect to the coordinates x and y) the toy-action:

$$S = 2x/\xi_x + 2y/\xi_y + \Gamma/T, \tag{54}$$

where the quantum and classical parts are related by a Fermi's Golden rule involving the density of localized states:

$$\nu \Gamma \mathcal{S} \approx 1, \tag{55}$$

where $\mathcal{S} = xy$ is the area. Equations (54) and (55) describe the process with the help of which an electron may hop from a site i to the site j within a disorder band with a constant density of localized states, ν. The first two terms in equation (54) represent the overlap between the two states separated by x and y. Because the system is disordered the energies of these states are different and a phonon has to be involved in the process of hopping. The thermal energy, Γ, is then determined with the help of equation (55) and corresponds to the average energy spacing between the localized states involved in the hopping: $\Gamma = 1/\nu\mathcal{S} = 1/\nu xy$. Introducing this value of Γ in equation (54) and minimizing the total action with respect to x and y yields the optimal hopping distances:

$$x_{\text{opt}} = \xi_x \ (T_M/T)^{1/3}, \quad y_{\text{opt}} = \xi_y \ (T_M/T)^{1/3},$$

where the parameter T_M reads:

$$T_M = 1/\nu\xi_x\xi_y. \tag{56}$$

This yields an optimal action: $S_{\text{opt}} \propto (T_M/T)^{1/3}$. The corresponding d.c. conductivity, $\sigma \propto \exp(-S_{\text{opt}})$, is the Mott law for variable-range hopping and reads:

$$\sigma_M(T) = \sigma_0(T) \exp[-(T_M/T)^{1/3}], \tag{57}$$

where σ_0 is a temperature-dependent pre-factor (the temperature-dependence arising from the phonon-scattering time which is a power-law). This law may be straightforwardly extended to any dimension. It may have a disorder-dependence through the constant DOS of the disorder band, ν:

$$\nu \propto \frac{N}{e^2/\kappa l} = \frac{\kappa}{e^2 b}, \qquad N \gg N_s, \tag{58a}$$

$$\nu \propto \frac{N}{e^2/\kappa l_s} = \frac{\kappa}{e^2 b}\,\frac{N}{N_s}, \quad N \ll N_s. \tag{58b}$$

This estimate was derived on the basis that we have N localized states per unit volume, each with an energy $e^2/\kappa l$ in the limit $N \gg N_s$ (charge excitations extend over segments between impurities, l) and $e^2/\kappa l_s$ in the limit $N \ll N_s$ (charge excitations have their length-scale, l_s). This shows that ν grows with N in the small impurity concentration case and saturates when N becomes larger than N_s. An additional dependence may come from ξ_x. In particular from equations (56) and (52), the parameter of the Mott-law for 2D quasi-1D systems reads:

$$T_M = \frac{r_s b N \ln(1/Nb^2)}{\nu \xi_y}, \qquad N \gg N_s, \tag{59a}$$

$$T_M = \frac{r_s b N_s \ln(1/N_s b^2)}{\nu \xi_y}, \quad N \ll N_s, \tag{59b}$$

where $N = 1/lb$ is the impurity-density, b the inter-wire distance and N_s has been defined by equations (10) and (17). It is interesting to notice, from equation (59a), that in the large impurity concentration case there is a linear dependence on N arising from ξ_x (ν is constant from equation (58a)). On the other hand, in the small impurity concentration case, equation (59b), ξ_x saturates and we have $T_M \sim 1/N$ from the $N-$dependence of ν, equation (58b). In this case, the conductivity increases with increasing disorder.

When the long-range Coulomb interaction is taken into account, the latter leads to a depletion of low-energy states in ν. As we have seen in Section 3, this depletion corresponds to a soft Coulomb gap $g(\varepsilon)$ in ν. Following Efros and Shklovskii, in the presence of this Coulomb gap, one has to replace equation (55) by:

$$g(\Gamma)\Gamma\mathcal{S} \approx 1. \tag{60}$$

With the help of equation (25), corresponding to a usual ES Coulomb gap shape, and the arguments above, the d.c. conductivity is the Efros-Shklovskii law for VRH:

$$\sigma_{ES}(T) = \sigma_0(T) \exp[-(T_{ES}/T)^{1/2}], \tag{61}$$

where the parameter T_{ES} depends on the specific Coulomb interaction among charge carriers:

$$T_{ES} = \frac{e^2}{\kappa\sqrt{\xi_x\xi_y}}. \tag{62}$$

Substituting the expression of the localization length, equation (52), this parameter reads:

$$T_{ES} = \frac{e^2}{\kappa}\sqrt{\frac{r_s N b \ln(1/Nb^2)}{\xi_y}}, \qquad N \gg N_s, \quad (63a)$$

$$T_{ES} = \frac{e^2}{\kappa}\sqrt{\frac{r_s N_s b \ln(1/N_s b^2)}{\xi_y}}, \qquad N \ll N_s. \quad (63b)$$

In the large impurity concentration case, $N \gg N_s$, this parameter increases with the impurity concentration, N, which leads to a decreasing conductivity as a function of disorder. This impurity-dependence originates from ξ_x. In the small impurity concentration case, $N \ll N_s$, ξ_x saturates and T_{ES}, as well as the corresponding conductivity, become disorder-independent.

At higher energies, the usual Coulomb gap of equation (25) crosses over to the unusual Coulomb gap of equation (27). The latter yields a conductivity that we denote as: σ_1, and reads:

$$\sigma_1(T) = \sigma_0(T)\exp\left[-(T_1/T)^{3/5}\right]. \quad (64)$$

The unusual exponent, i.e. 3/5 instead of 1/2, is related to the unusual Coulomb gap of equation (27). Furthermore, the parameter T_1 reads:

$$T_1 = \frac{e^2}{(b\kappa^2\kappa_x\xi_x\xi_y)^{1/3}}. \quad (65)$$

Substituting the expression of ξ_x, equation (52), and of κ_x, equation (22), yields:

$$T_1 = \frac{e^2}{\kappa b}\, Nb^2 \left[\frac{r_s b \ln(1/Nb^2)}{\xi_y}\right]^{1/3}, \quad N \gg N_s, \quad (66a)$$

$$T_1 = \frac{e^2 b}{\kappa}\left[\frac{r_s b N^2 N_s \ln(1/N_s b^2)}{\xi_y}\right]^{1/3}, \quad N \ll N_s. \quad (66b)$$

The T_1 parameter depends on disorder through both ξ_x and κ_x, in the large-impurity concentration regime. On the other hand, ξ_x saturates in the small impurity concentration case and the disorder dependence of T_1 originates then only from κ_x. In both cases this disorder-dependence of T_1 yields a conductivity, σ_1, which decreases with increasing disorder.

We then focus on the peculiar 2/3 Coulomb-gap sector in the small impurity case, cf. equation (31). This Coulomb-gap yields:

$$\sigma_2(T) = \sigma_0(T)\exp[-(T_2/T)^{5/11}], \quad N \ll N_s. \quad (67)$$

The unusual exponent, i.e. 5/11 instead of 1/2, is again related to the unusual Coulomb gap (here $\propto \varepsilon^{2/3}$) of equation (31). Furthermore, the parameter T_2 reads:

$$T_2 = e^2\left[\frac{\alpha r_D}{\xi_x^3\ \xi_y^3}\right]^{1/5}, \quad N \ll N_s. \quad (68)$$

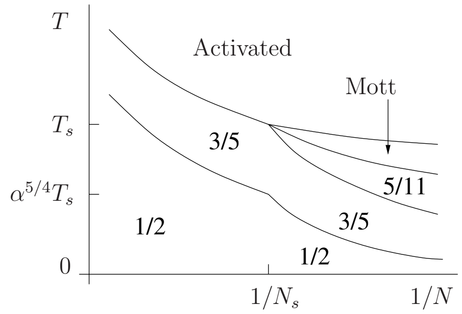

Fig. 4. The generic phase diagram of a 2D quasi-1D system as a function of temperature T and the inverse average impurity concentration $1/N$ [$T_s \propto E_s$ the soliton energy]. This phase diagram displays the VRH laws with exponents 1/2, 3/5 and 5/11 as well as Mott law and activation at higher temperatures.

Substituting the expression for ξ_x, equation (52), this parameter reads:

$$T_2 = e^2\left[\frac{\alpha r_s^3 b^4 N_s^3 \ln^3(1/N_s b^2)}{\xi_y^3}\right]^{1/5}, \quad N \ll N_s. \quad (69)$$

This parameter, as well as the corresponding conductivity, σ_2, are disorder-independent.

Finally, at high energies (within the constant DOS ν), these VRH laws cross-over to the so called nearest-neighbor hopping (NNH) or activated law, see reference [9] for a review. The latter reads:

$$\sigma_{NNH}(T) = \sigma_0(T)\exp[-E_{NNH}/T], \quad (70)$$

where the NNH energy reads:

$$E_{NNH} = e^2 bN/\kappa, \quad N \gg N_s, \quad (71a)$$

$$E_{NNH} = e^2 bN_s/\kappa, \quad N \ll N_s. \quad (71b)$$

These results are summarized on the temperature − impurity-concentration phase-diagram of Figure 4. The crossover lines between the different laws in Figure 4 are determined with the help of the following arguments.

The expression of ε_1, equation (29), determines the crossover temperature lines $T_1(N)$ between the ES and the 3/5−laws [$\varepsilon_1 = T_{c1}(T_{ES}/T_{c1})^{1/2}$] with T_{ES} given by equation (63). This crossover line reads

$$T_{c1} = \frac{e^2}{\kappa b}\,(Nb^2)^{7/2}\sqrt{\frac{\xi_y}{r_s b \log(\frac{1}{Nb^2})}}, \quad N \gg N_s, \quad (72a)$$

$$T_{c1} = \frac{e^2}{\kappa b^2}\,(Nb^2)^4\sqrt{\frac{\xi_y}{r_s b N_s \log(\frac{1}{N_s b^2})}}, \quad N \ll N_s. (72b)$$

In the case of large impurity concentration, the expression of Δ_1, equation (28) determines the total width of the Coulomb gap and depends on the constant DOS ν. This width determines the high temperature crossover line $T_{c2}(N)$, [$\Delta_1 = T_{c2}(T_1/T_{c2})^{3/5}$], between the 3/5-law and nearest-neighbor-hopping, cf. equation (71a), for $N \gg N_s$.

With the help of equation (64), $T_{c2}(N)$ reads:

$$T_{c2}(N) = \frac{e^2}{\kappa b}\, N b^2 \left[\frac{e^2 \nu b}{\kappa}\right]^{5/4} \sqrt{\frac{\xi_y}{r_s b \ln(1/Nb^2)}}, \quad N \gg N_s. \qquad (73)$$

Using the estimation for ν, equation (58a), equation (73) may be re-expressed as:

$$T_{c2}(N) = \frac{e^2}{\kappa b}\, N b^2 \sqrt{\frac{\xi_y}{r_s b \ln(1/Nb^2)}}, \; N \gg N_s. \qquad (74)$$

From equations (72a) and (73), we see that: $T_{c2}(N)/T_{c1}(N) = 1/(b^2 N)^{5/2} \gg 1$, so that at the boundary $N \sim N_s$, the corresponding crossover temperatures are separated by the large dimensionless factor $(l_s/b)^{5/2} = 1/\alpha^{5/4}$. Moreover, we define $T_s \equiv T_{c2}(N_s)$ which reads, up to a logarithmic factor:

$$T_s \equiv T_{c2}(N_s) \propto E_s, \qquad (75)$$

where: $E_s = e^2/\kappa l_s$, is the creation energy of a soliton. With these notations we have:

$$T_{c1}(N_s) \propto \alpha^{5/4} T_s \ll T_s. \qquad (76)$$

We could also show that in this large impurity concentration case there is no room for the Mott law of equation (59a), i.e. the crossover line from the 3/5-law to the Mott law coincides with the crossover line from the Mott law to the activated- or NNH- law.

In the small impurity concentration case we have already given the expression for the T_1-line in equation (72b), which matches smoothly equation (72a) at $N \sim N_s$. On the other hand, the T_2-line splits into three crossover lines thereby opening two sectors for the new 5/11-law and the Mott law. The energy ε_2, equation (32), determines the temperature crossover-line: $T^a_{c2}(N)$, $[\varepsilon_2 = T_{c2}(T_1/T_{c2})^{3/5}]$, between the 3/5-law and the 5/11-laws for $N \ll N_s$. With the help of equation (66b) for the parameter T_1 and equation (32), $T^a_{c2}(N)$ reads:

$$T^a_{c2}(N) = \frac{e^2}{\kappa l_s} \left[\frac{N}{N_s}\right]^{11/4} \sqrt{\frac{\xi_y}{r_s b \log(1/N_s b^2)}}, \quad N \ll N_s. \qquad (77)$$

which matches smoothly equation (74) at $N \sim N_s$ and decreases more abruptly with $1/N$ for $N \ll N_s$.

At the next crossover-line: $T^b_{c2}(N)$, the 5/11−law crosses over to the Mott VRH law with the parameter equation (59b). This crossover-line reads:

$$T^b_{c2}(N) = \frac{e^2}{\kappa l_s} \left[\frac{N}{N_s}\right]^{5/2} \sqrt{\frac{\xi_y}{r_s b \log(1/N_s b^2)}}, \quad N \ll N_s, \qquad (78)$$

which matches smoothly equations (74) and (77) at $N \sim N_s$. Finally, at higher temperatures a crossover-line bridges the Mott law of equation (59b) with the NNH law of equation (71b):

$$T^c_{c2}(N) = \frac{e^2}{\kappa l_s} \left[\frac{N}{N_s}\right]^{1/2} \sqrt{\frac{\xi_y}{r_s b \log(1/N_s b^2)}}, \quad N \ll N_s, \qquad (79)$$

which matches smoothly equations (74), (77) and (78) at $N \sim N_s$.

6 Non-linear VRH laws

This section deals with the hopping laws of the quasi-1D Mott-Anderson insulators in the non-linear response regime. Our results depend crucially on the Coulomb gap shapes derived in Section 3 and on the impurity-dependence of the longitudinal localization length derived in Section 4. Our arguments follow closely those of the linear-response regime of Section 5. Notice that all transport laws below, and especially the parameter characteristic of these laws, are given up to a numerical coefficient of the order of unity. Notice also that our present arguments do not allow us to determine the pre-factor of the current; we therefore focus only on its main exponential dependence.

The VRH laws derived in Section 5 are valid in the linear response (or ohmic) regime where the current is linear in the applied electric field, $j = \sigma(T)\mathcal{E}$. At a given temperature, upon increasing the electric field, a crossover should take place, below the threshold for the sliding of the electronic crystal, to a non-linear regime. In the frame of doped semiconductors, such a transition has been studied in reference [37] by extending the Mott argument presented in the previous subsection on the linear VRH laws. Such an extension amounts to replace, in Fermi's Golden rule of equation (55), the hopping energy Γ by the energy provided by the electric field, $e\mathcal{E}r$, during the motion of an electron along a distance r from the initial localized state. In d dimensions, the optimal hopping distance is therefore given by: $r_{\text{opt}} = 1/(\nu e\mathcal{E})^{1/1+d}$. Substituting this value in the tunneling probability $\propto\exp(-2r/\xi)$ yields:

$$j_{MS}(\mathcal{E}) \sim \exp\left[-\left(\frac{\mathcal{E}_{MS}}{\mathcal{E}}\right)^{\frac{1}{1+d}}\right], \;\; \mathcal{E}_{MS} = \frac{1}{\nu e \xi^{1+d}}. \qquad (80)$$

where the index MS refers to Mott-Shklovskii, see also reference [38]. The crossover between the linear and non-linear regimes takes place when: $(\mathcal{E}_{MS}/\mathcal{E})^{1/1+d} = (T_M/T)^{1/1+d}$, where the parameter of the Mott law: $T_M = 1/\nu\xi^d$, in d-dimensions. At a given temperature, the threshold field for the non-ohmic regime is given by: $\mathcal{E}_c = (\mathcal{E}_M/T_M)T$, that is:

$$\mathcal{E}_c = \frac{T}{e\xi}, \qquad (81)$$

in all dimensions. This returns us to the arguments of the Introduction, see equation (4) and discussion around it.

It is straightforward to generalize such results in the presence of a Coulomb gap. We need simply to replace the constant DOS, ν, in Fermi's Golden rule by the corresponding Coulomb gap: $g(\varepsilon)$, evaluated at the energy of the charge excitation: $\varepsilon = eEr$. The generic form of the Coulomb-gap (CG) in isotropic system is given by equation (24). This leads to an optimal hopping distance independent of dimensionality: $r_{\mathrm{opt}} = (e^2/\kappa e\mathcal{E})^{1/2}$. The corresponding current is therefore given by:

$$j_0(\mathcal{E}) \sim \exp[-(\mathcal{E}_0/\mathcal{E})^{\frac{1}{2}}], \quad \mathcal{E}_0 = \frac{e^2}{\kappa e\xi^2}. \tag{82}$$

Strictly speaking, equation (82) is valid for $d > 1$. In $d = 1$, there is a logarithmic Coulomb gap: $g(\varepsilon) = \nu/\log(e^2Nb/\kappa|\varepsilon|)$, see reference [14] for a similar derivation of this law in the ohmic regime. For the 1D case, the parameter of equation (82) is therefore given by:

$$\mathcal{E}_0 = \frac{\log[(e^2Nb/\kappa)\sqrt{\nu/\mathcal{E}}]}{\nu e\xi^2}, \quad d = 1, \tag{83}$$

in the logarithmic approximation where the energy of the charge excitation has been taken equal to $\varepsilon = e\mathcal{E}r_{\mathrm{opt}}$ and $r_{\mathrm{opt}} = 1/\sqrt{\nu e\mathcal{E}}$ is the optimal hopping distance in 1D.

These results may be extended to quasi-1D systems. For simplicity, we assume that the *electric field is parallel to the chains*. More general results with the two components of the field can be derived in the same way. We also focus on the 2D case. Then, Fermi's Golden rule of equation (55) is generalized to: $\nu(e\mathcal{E}_x x)xy \sim 1$. The tunneling action is anisotropic and takes the usual form $S = 2x/\xi_x + 2y/\xi_y$. Substituting the $x-$component of the hopping length in this action and minimizing the resulting expression with respect to y yields to the optimal hopping distances. Substituting the latter in the action we finally obtain a current:

$$j_{MS}(\mathcal{E}) \sim \exp\left[-\left(\frac{\mathcal{E}_{MS}}{\mathcal{E}}\right)^{\frac{1}{3}}\right], \quad \mathcal{E}_{MS} = \frac{1}{\nu e\xi_x^2\xi_y}. \tag{84}$$

Substituting the expression for the longitudinal localization length, equation (52), this yields:

$$\mathcal{E}_{MS} = \frac{r_s^2b^2N^2\ln^2(1/Nb^2)}{e\nu\xi_y}, \qquad N \gg N_s,$$

$$\mathcal{E}_{MS} = \frac{r_s^2b^2N_s^2\ln^2(1/N_sb^2)}{e\nu\xi_y}, \quad N \ll N_s.$$

Substituting the expressions for ν, equation (58), yields the explicit $N-$dependence of the parameter:

$$\mathcal{E}_{MS} = \frac{e^2}{\kappa b}\,(Nb^2)^2\,\frac{r_s^2\ln^2(\frac{1}{Nb^2})}{e\xi_y}, \qquad N \gg N_s, \tag{86a}$$

$$\mathcal{E}_{MS} = \frac{e^2}{\kappa b}\,\frac{N_s}{N}\,\frac{r_s^2(N_sb^2)^2\ln^2(\frac{1}{N_sb^2})}{e\xi_y}, \quad N \ll N_s. \tag{86b}$$

The next step consists, as for the isotropic system, in introducing the Coulomb interaction, which amounts to replace the constant DOS, ν, by the Coulomb gaps determined in Section 3. For the linear Coulomb gap of equation (25) the non-linear law follows equation (82) with a modified parameter:

$$\mathcal{E}_0 = \frac{e}{\kappa}\left[\frac{1}{\xi_x\xi_y^{\frac{1}{3}}}\right]^{\frac{3}{2}}. \tag{87}$$

Substituting the expression for the longitudinal localization length, equation (52), this parameter reads:

$$\mathcal{E}_0 = \frac{e}{\kappa}\,\left[\frac{r_sbN\ln(1/Nb^2)}{\xi_y^{1/3}}\right]^{3/2}, \quad N \gg N_s, \tag{88a}$$

$$\mathcal{E}_0 = \frac{e}{\kappa}\,\left[\frac{r_sbN_s\ln(1/N_sb^2)}{\xi_y^{1/3}}\right]^{3/2}, \; N \ll N_s. \tag{88b}$$

On the other hand, the anomalous quadratic CG of equation (27) leads to a 3/5$-$law in the non-linear regime:

$$j_1(\mathcal{E}) \sim \exp[-(\mathcal{E}_1/\mathcal{E})^{3/5}], \tag{89}$$

where the parameter reads:

$$\mathcal{E}_1 = \frac{e}{\kappa}\left[\frac{\kappa}{\kappa_x\xi_x^4\xi_yb}\right]^{1/3}. \tag{90}$$

Substituting the expression for the longitudinal localization length, equation (52), this parameter reads:

$$\mathcal{E}_1 = \frac{e}{\kappa b^2}\,(Nb^2)^2\,\left[\frac{r_s^4b\ln^4(\frac{1}{Nb^2})}{\xi_y}\right]^{\frac{1}{3}}, \quad N \gg N_s, \tag{91a}$$

$$\mathcal{E}_1 = \frac{e}{\kappa b^2}\,(NN_s^2b^6)^{\frac{2}{3}}\left[\frac{r_s^4b\ln^4(\frac{1}{Nb^2})}{\xi_y}\right]^{\frac{1}{3}}, N \ll N_s. \tag{91b}$$

Finally, in the small impurity concentration case, the 2/3$-$Coulomb gap of equation (31) leads to a 5/11$-$law for the current:

$$j_2(\mathcal{E}) \sim \exp[-(\mathcal{E}_2/\mathcal{E})^{5/11}], \quad N \ll N_s, \tag{92}$$

with a parameter

$$\mathcal{E}_2 = \frac{e}{\kappa}\left[\frac{\alpha r_D}{\xi_x^8\;\xi_y^3}\right]^{1/5}, \quad N \ll N_s. \tag{93}$$

Substituting the expression for the longitudinal localization length, equation (52), in the small impurity case, this parameter reads:

$$\mathcal{E}_2 = \frac{e}{\kappa l_s^2}\,\left[\frac{r_s^8b^3\ln^8(1/N_sb^2)}{\xi_y^3}\right]^{1/5}, \quad N \ll N_s. \tag{94}$$

From equations (86), (88), (91) and (94) for the parameters and the expressions of the corresponding currents, the non-monotonous behavior of j as a function of N is rather clear.

We shall not detail the field crossover lines between the various laws. This can be done exactly in the same way as for the linear-response regime case. Rather than that we would like to point out that the phase-diagram, electric-field vs. impurity concentration, may be obtained from the one of the linear law by assuming that the electric field gives rise to an "effective" temperature, $T_{\mathcal{E}}$. This effective temperature reads:

$$T_{\mathcal{E}} \;\equiv\; e\,\mathcal{E}\,x_{\text{opt}}, \tag{95}$$

where x_{opt} is the optimal hopping distance along a wire (recall that the electric field is assumed to be parallel to the chains). For a given non-linear (NL) law of parameter $\mathcal{E}_{NL}$ and exponent γ_{NL}, the optimal hopping distance along the wires is defined as:

$$x_{\text{opt}}(N,\mathcal{E}) = \xi_x\,\left(\mathcal{E}_{NL}/\mathcal{E}\right)^{\gamma_{NL}}, \tag{96}$$

and depends on the applied electric field and, eventually, the impurity concentration. The phase diagram, $T_{\mathcal{E}}(N)$ vs. N, therefore has a non-trivial re-scaling via the N-dependence of $T_{\mathcal{E}}$.

7 Conclusion

We attempted to construct a semi-phenomenological theory of variable-range hopping for 2D quasi-1D systems such as arrays of quantum wires in the Wigner-crystal regime. We have closely followed the phenomenological arguments of Mott, Efros and Shklovskii to derive the Coulomb gap shapes, Section 3, as well as the main exponential dependence of the transport laws in the linear, Section 5, and non-linear, Section 6, response regimes. Our approach has been supplemented with some microscopic arguments necessary to derive the impurity-dependence of the longitudinal localization length, Section 4. Both Coulomb gap shapes and transport laws were found to have rather unusual features with respect to known results in the field of disordered semiconductors [9]. These unusual features arise because of the non-trivial dielectric properties of the systems under consideration, Section 2. They are two-fold: a non-monotonous dependence of the conductivity or current as a function of disorder and a highly non-universal exponent γ. In the linear response regime, the richness of exponents is displayed on the phase diagram of Figure 4. Despite the fact that some exponents may be close to each-other ($1/2 = 0.50$, $3/5 = 0.60$, $5/11 = 0.45$, $1/3 = 0.33$), the corresponding monotonicity of the conductivity as a function of disorder enables further discrimination between the various laws. This is schematically displayed in Figure 5. Moreover, 2D layers of wire arrays are experimentally accessible, to our knowledge [15], but we are unaware of systematic transport experiments in the strongly localized regime for such systems. Our theory is therefore only of predictive nature. We hope, however, that it will be of some interest to both theorists and experimentalists working in the field.

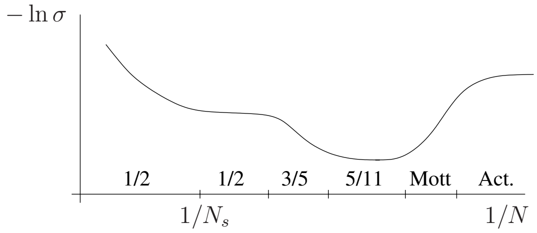

Fig. 5. The logarithm of the resistivity, in the linear response regime, versus the inverse average impurity concentration, $1/N$, for a 2D quasi-1D system at a temperature: $T \ll \alpha^{5/4} T_s$, where $T_s \propto E_s$, see Figure 4. The ES (Eq. (61)), 3/5 (Eq. (64)), 5/11 (Eq. (67)), Mott (Eq. (57)) and activation (Eq. (70)) law succeed each-other with decreasing N.

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Appendix A: 2D model of interrupted strands

As an alternative to the use of equations (18) and (19) the "disorder averaged" electrostatic potential may be derived following the results of the interrupted strand model, see reference [31] of Rice and Bernasconi for the 3D case. In this approximation, we *assume* that the dielectric constant along the chains is given by equation (22). This equation shows that for $L_x/r_D = l/r_D \gg 1$, the contribution of the electronic part along the chains is much larger than the transverse part as well as the host lattice dielectric constant ($\kappa_y = \kappa \ll \kappa_x$). The problem therefore reduces to determine the electrostatic potential of a charge carrier in a layer of a quasi-one dimensional system whose longitudinal dielectric constant is much larger than the dielectric constant of the surrounding media. Keldysh [39] has solved a similar problem for an isotropic layer. In the present case, the Fourier transform of the interaction potential at $(\mathbf{r}, z)$ due to a point charge at the origin of the layer is found by usual methods of electrostatics and reads:

$$U(\mathbf{q}, z) = \frac{4\pi e^2}{\breve{q} + q}\,\frac{\cosh\left(\breve{q}b/2 + \delta\right)}{\sinh\left(\breve{q}b + 2\delta\right)}\,\exp\left(q\left(b/2 - z\right) + \delta\right), \tag{A.1}$$

where $\mathbf{q} = (q_x, q_y)$ is the two-dimensional reciprocal vector of $\mathbf{r}$, $\breve{q}^2 = \kappa_x q_x^2 + q_y^2$ and δ is given by:

$$\delta = \frac{1}{2}\log\left(\frac{\breve{q} + q}{\breve{q} - q}\right). \tag{A.2}$$

At large distances $\breve{q}b \ll 1$, i.e. for $x \gg l$, that will be of interest to us in the following, and for $z = 0$, equation (A.1) reduces to equation (18) with the following dielectric function:

$$\varepsilon_2(\mathbf{q}) \approx \kappa \left[1 + \frac{\kappa_x}{\kappa}\frac{bq_x^2}{q}\right]. \quad (A.3)$$

This expression is equal to equation (19) in the limit $\tilde{q} = q_x \ll L_x^{-1}$, with $\kappa_x \approx \kappa(L_x/r_D)^2$ and $r_D \approx b$. This shows that in this limit, both approaches are equivalent.

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